COMPLEMENTED SUBSPACES AND Λ SYSTEMS IN BANACH SPACES

BY

WILLIAM J. DAVIS*, DAVID W. DEAN**, AND IVAN SINGER

ABSTRACT

In this paper we study the problems of existence of noncomplemented subspaces and Lozinsky-Kharshilade systems [10] in Banach spaces not isomorphic to Hilbert spaces.

1. Introduction

In [1] Banach asked whether a B-space could have a noncomplemented subspace. This was answered for some concrete spaces like the L_p, (l_p) $(1 \leqq p \neq 2)$, (c_0) and (m) in [13, 17, 4(p.553)]. All $C(H)$ spaces, H infinite compact Hausdorff have such subspaces since they contain copies of (c_0). In the same way, all universal spaces for separable spaces have noncomplemented subspaces.

One expects that almost all B-spaces have noncomplemented subspaces. Thus the structure of Hilbert space is especially well known, partly since every subspace is complemented. The converse question, whether a space, every subspace of which is complemented is isomorphic to Hilbert space, is still unsolved. It remains unsolved even if one assumes the existence of a constant K such that every subspace admits a projection with norm $\leqq K$.

In §2 of this paper, sufficient conditions are given for a B-space to have a noncomplemented subspace. In certain cases, the conditions are necessary. In §3 we study the related notion of Λ-system (Lozinski-Kharshiladze system [10]), showing that except for spaces isomorphic to Hilbert space, the usual concrete separable Banach spaces have such systems. Finally, in §4, we raise some problems.

We wish to express our gratitude to Professor Joram Lindenstrauss for reading the manuscript and making valuable remarks.

2. Complemented subspaces

Following F. J. Murray [13], define the projection constant $\lambda(X)$ for a subspace X of a Banach space E to be the infimum of the set of norms of projections from E onto X, or $\lambda(X) = \infty$ if X is not complemented in E.

Received April 26, 1968 and in revised form June 30, 1968.

* Supported by N.S.F. Grant number GP–6152

** Supported by N.S.F. Grant number GP–2242

We shall now prove that if a separable Banach space has finite dimensional subspaces with arbitrarily large projection constants, then the space has a noncomplemented subspace. It would be desirable to have a converse to this statement, but we shall see that only partial converses are available.

We shall need the following lemma at several points in the paper. We set $\lambda_f(X) = \sup\{\lambda(F) \mid F \text{ is a finite dimensional subspace of } X\}$.

LEMMA 1. If $E = G \oplus Y$, $\dim G < \infty$ and $\lambda_f(Y) < \infty$, then $\lambda_f(E) < \infty$.

Proof. If Z is a closed subspace of E and if $X \supset Z$ with $\dim X/Z = N < \infty$, then there exists a projection Q_ε of X onto Z with norm $\leqq 2^N + \varepsilon$ (see e.g. [7]). Thus, if $P: E \to X$ is a projection, then $Q_\varepsilon \circ P: E \to Z$, and it follows that $\lambda(Z) \leqq 2^N \lambda(X)$. Now, let $\varepsilon > 0$ and $P_\varepsilon: E \to Y$ be a projection with $\| P_\varepsilon \| < 2^N + \varepsilon$ (where $N = \dim G$). Let F be an arbitrary finite dimensional subspace of E, and let $F \subset F' = (I - P_\varepsilon)(E) \oplus P_\varepsilon(F)$. Since $\dim P_\varepsilon(F) \leqq \dim F < \infty$, there is a projection Q from Y onto $P_\varepsilon(F)$ with $\| Q \| < \lambda_f(Y) + \varepsilon$. Therefore, $(I - P_\varepsilon) + Q \circ P_\varepsilon$ is a projection of E onto F'. Therefore, $\lambda(F') \leqq 2^N + 1 + 2^N \lambda_f(Y)$, and since $\dim F'/F \leqq N$, we see that $\lambda(F) \leqq 2^N(2^N + 1 + 2^N \lambda_f(Y))$, and so $\lambda_f(E) < \infty$.

We are now ready for the first theorem.

THEOREM 1. *Let E be a Banach space such that $\lambda_f(E) = \infty$. Then E has a noncomplemented subspace, F. The subspace F has a Schauder decomposition into finite-dimensional subspaces.*

Proof. Let X_1 be a finite dimensional subspace of E with $\lambda(X_1) \geqq 1$. Let $E_1 = X_1$. Choose $(f_1^1, \cdots, f_{n_1}^1) \subset E_1^*$ so that $\|f_j^1\| = 1$, and let $(g_1, \cdots, g_{n_1}) \subset E^*$ be Hahn-Banach extensions of the f_j^1's, and where these are chosen so that $[\bigcap_{j=1}^{n_1} g_j^{-1}([-1,1])] \cap E_1$ is contained in the 2 ball of E_1. Let $Y_1 = \bigcap_{j=1}^{n_1} g_j^{-1}(0)$. Then $E_1 \cap Y_1 = \{0\}$ and the natural projection of $E_1 \oplus Y_1$ onto E_1 has norm $\leqq 2$. By lemma 1, choose $E_2 \subset Y_1$ so that $\dim E_2 < \infty$ and $\lambda(E_2) \geqq 2$. As above, we find $(g_{n_1+1}, \cdots, g_{n_2})$ in E^* with $\| g_j \| = 1$ and so that $[\bigcap_{j=1}^{n_2} g_j^{-1}([-1,1])] \cap (E_1 \oplus E_2)$ is in the 2 ball of $E_1 \oplus E_2$. With $Y_2 = \bigcap_{j=1}^{n_2} g_j^{-1}(0)$, we have $Y_2 \subset Y_1$, codim $Y_2 \leqq n_2$ and the natural projection of $E_1 \oplus E_2 \oplus Y_2$ onto $E_1 \oplus E_2$ has norm $\leqq 2$. Proceeding in this way, we obtain (E_n) and (Y_n) such that $\lambda(E_n) \geqq n$, $Y_{n+1} \subset Y_n$ and the natural projections of $E_1 \oplus \cdots \oplus E_n \oplus Y_n$ onto $E_1 \oplus \cdots \oplus E_n$ have norm $\leqq 2$.

Now define $F = \sum_{n=1}^{\infty} E_n = \{ \sum e_n \mid e_n \in E_n \text{ and } \sum e_n \text{ converges in } E\}$. By standard arguments, F is a closed subspace of E and has the Schauder decomposition (E_n). Let P_n be the natural projection of F onto $E_1 \oplus \cdots \oplus E_n$. Then if P is a projection of E onto F, $(I - P_{n-1})P_n P$ is a projection of E onto E_n, and we see that $\lambda(E_n) \leqq 6 \, \| P \|$. This is impossible, so F is noncomplemented.

REMARK 1. The first part of the argument in the proof above uses the technique found in [3], and yields the following: For any Banach space E, any finite dimensional subspace X of E, and any $\varepsilon > 0$, there exists a subspace Y of finite codimension in E such that $X \cap Y = \{0\}$ and the natural projection of $X \oplus Y$ onto X has norm $\leqq 1 + \varepsilon$.

REMARK 2. The final part of the argument in the proof above also shows that if P is any projection of E onto $E_1 \oplus E_2 \oplus \cdots \oplus E_n \oplus B$ where $B \subset E_{n+1}$, then $\| P \| \geqq n/6$. We shall use this fact in the proof of Theorem 3 below.

The following result is similar to a result of Lindenstrauss [11], and its proof is the same. It furnishes a partial converse to Theorem 1.

THEOREM 2. *If E is a reflexive Banach space, and if $\lambda_f(E) < \infty$, then every subspace of E is complemented and admits a projection with norm $\leqq \lambda_f(E)$.*

From Theorems 1 and 2 we infer:

COROLLARY. *If E is a separable reflexive Banach space, and all subspaces of E which admit Schauder decompositions into finite dimensional subspaces are complemented, then there is a constant $K \geqq 1$ such that every subspace of E admits a projection of norm $\leqq K$.*

In Lindenstrauss [11], the projection of Theorem 2 is constructed as the weak operator limit of a sequence of projections, P_n, satisfying $(\| P_n \|)$ bounded and $P_{n+1}(E) \supset P_n(E)$ for all n. In general, for separable E, such a sequence of projections need not converge even in weak operator topologies, as shown by the following example, communicated to us by V. I. Gurarii and M. I. Kadec:

Let X be a subspace of $C([0,1])$ such that X is noncomplemented in $C([0,1)]$, and X is isomorphic to $C([0,1])$ (see, e.g. [6]). Let $(x_n) \subset X$ be the image under the isomorphism of the usual Schauder basis $(z_n) \subset C([0,1])$. Then, if $C_n = [x_1, \cdots, x_n]$, by virtue of [12], Corollary 6.2 and Lemma 2.1 there exist projections (P_n) such that P_n: $C([0,1]) \to C_n \subset X$, $(\| P_n \|)$ bounded, and $[C_n] = X$ not complemented in $C([0,1])$.

3. Λ-systems

In view of Theorem 1, the following definition is natural and useful in the study of projections onto finite dimensional subspaces.

DEFINITION. *A linearly independent sequence (x_n) in a Banach space E is called a sub Λ system [sub Γ system] if*

$$\lambda([x_1, \cdots, x_n]) \overset{n}{\to} \infty \ [\ \sup_n \lambda([x_1, \cdots, x_n]) = \infty\ ].$$

The sequence is a Lozynski-Kharshiladze system or Λ system [resp. Γ system] if also $[x_n] = E$.

From Theorem 1 it is clear that if E has a sub Γ system, then E has a noncomplemented subspace. However, spaces with Λ systems may easily be constructed in which the system has a subsequence spanning a complemented subspace.

S. M. Lozynski and F. I. Kharshiladze have proved (see [14], appendix 3) that the sequence $x_n(t) = t^{n-1}$ $(t \in [0,1],\ n = 1,2,\cdots)$ is a Λ system in $C([0,1])$. Using Sobczyk's construction [17] for a noncomplemented subspace of l_p or c_o, one easily constructs a sub Γ system as below. The problem, given a noncomplemented subspace construct a sub Γ system, remains open.

Consider $l_p = (\sum_{n=1}^{\infty} l_n^p)_p$ [11]. Define $\bar{c}(l_n^p) = \sup\{\lambda(X) \mid X \subset l_n^p\}$. Then $\lim_n \bar{c}(l_n^p) = \infty$ [17]. In l_n^p choose $x_1^n, \cdots, x_{k_n}^n$ so that $\lambda([x_1^n, \cdots, x_{k_n}^n]) \to \infty$. Then the sequence $(x_1^1, \cdots, x_{k_1}^1, x_1^2, \cdots)$ is a sub Γ system in l_p (since for any projection $P: l_p \to [x_1^1, \cdots, x_{k_n}^n]$ we have $\| P \| \geqq \| P \mid l_n^p \| \geqq \lambda([x_1^n, \cdots, x_{k_n}^n])$). A similar argument yields such systems in c_o.

Theorem 3. *Let E be a Banach space. The following are equivalent.*

(*a*) *There is a sequence (X_n) of finite dimensional subspaces of E such that $\lambda(X_n) \to \infty$.*

(*b*) *E has a sub Γ system.*

(*c*) *E has a sub Λ system.*

(*d*) *E has a Λ system (if E is separable).*

Proof. (a) says that $\lambda_f(E) = \infty$, so if we let E_n (from the proof of Theorem 1) have basis $(e_1^n, \cdots, e_{p_n}^n)$, from Remark 2 above it follows that the sequence $(e_1^1, \cdots, e_{p_1}^1, e_1^2, \cdots)$ is a sub Λ system, so (a) implies (c). That (c) implies (b) implies (a) is clear, and (d) implies (a) by the definition of a Λ system. We now show that (c) implies (d).

Let $\{x_1, \cdots\}$ be a sub Λ system and let $F = [x_n]$. Choose a sequence $\{y_n\}$ in E, not meeting F, such that $[x_i, y_j] = E$ and $\{x_i, y_j\}$ is a linearly independent set. Choose a projection P_m from E onto $[y_1, \cdots, y_m]$ such that $P_m(x_n) = 0$ for each n. This may be done using the linear independence, that $F \cap [y_1, \cdots, y_m] = 0$, and the Hahn-Banach theorem. Let $Q_{n,k}$ be a projection from E onto

$$[x_1, \cdots, x_n, y_1, \cdots, y_k]$$

and let $R_k = I - P_k$. Then $R_k Q_{n,k}$ is a projection onto $[x_1, \cdots, x_n]$. Since $\lambda([x_1, \cdots, x_n]) \leqq \| Q_{n,k} \| \, \| R_k \|$, one has $\| Q_{n,k} \| \geqq \lambda([x_1, \cdots, x_n]) / \| R_k \|$. Then choose n_k such that $n \geqq n_k$ implies $\lambda([x_1, \cdots, x_n]) \geqq \| R_k \| n$. Then the sequence $x_1, \cdots, x_{n_1}, y_1, x_{n_1+1}, \cdots, x_{n_2}, y_2, \cdots$, i.e. the sequence $(z_n) \subset E$ defined by

$$z_n = \begin{cases} x_{n-k+1} & \text{for } n_{k-1} + k \leqq n \leqq n_k + k - 1 \quad (k = 1,2,\cdots) \\ y_k & \text{for } n = n_k + k \quad (k = 1,2,\cdots) \end{cases}$$

where $n_0 = 0$, is a Λ system in E. Indeed, $(z_n) = (x_i, y_j)$ is linearly independent and $[z_n] = E$. Furthermore, if $n_{k-1} + k \leqq n \leqq n_k + k - 1$, and if Q is an arbitrary projection of E onto $[z_1,\cdots,z_n]=[x_1,\cdots,x_{n-k+1},y_1,\cdots,y_{k-1}]$, then Q is a $Q_{n-k+1,k-1}$ and so by the choice of n_{k-1},

$$\|Q\| = \|Q_{n-k+1,k-1}\| \geqq \frac{\lambda([x_1,\cdots,x_{n-k+1}])}{\|R_{k-1}\|} \geqq \frac{\|R_{k-1}\|(n-k+1)}{\|R_{k-1}\|}$$
$$= n-k+1 \geqq n_{k-1}+1.$$

Similarly, if Q is an arbitrary projection of E onto

$$[z_1,\cdots,z_n]=[x_1,\cdots,x_{n-k},y_1,\cdots,y_k],$$

then

$$\|Q\| = \|Q_{n-k,k}\| \geqq \frac{\lambda([x_1,\cdots,x_{n-k}])}{\|R_k\|} \geqq \frac{\|R_k\|(n-k)}{\|R_k\|} = n_k,$$

which completes the proof.

Corollary 2. *If a separable Banach space E has a subspace F with a Λ system, then the space E has a Λ system. In particular every Λ system of F extends to a Λ system of E.*

Proof. Observe that every Λ system $(x_n) \subset F \subset E$ is a sub Λ system of E, since for any projection Q of E onto $[x_1,\cdots,x_n], Q|F$ is a projection of F onto $[x_1,\cdots,x_n]$ and $\|Q|F\| \leqq \|Q\|$. Thus by Theorem 3 and its proof, (x_n) can be extended by a Λ system of E.

From this corollary and the remark before Theorem 3 it follows that, in particular, the spaces L_p, $C(H)$ (H compact metric) and all other universal spaces for separable spaces have Λ systems. The existence of Λ systems for L_p spaces was obtained by a different method by M. I. Kadec [10].

The converse of the second statement of the corollary is not true, i.e., a subsequence of a Λ system need not be a Λ system in its closed linear span—it may even be basic. Moreover, it is not hard to see that every linearly independent complete sequence in a separable Banach space F can be extended to a Λ system of a suitable superspace E. It is not known whether some subsequence of every Λ system is basic. The answer to this is affirmative for the known concrete Λ systems.

Corollary 3. *If E has a non-reflexive subspace with an unconditional basis, then E has a Λ system.*

Proof. Such an E has [8] a subspace isomorphic to c_0 or l_1. The result follows from Corollary 2.

4. Remarks and unsolved problems

In Theorem 1 there remains the question of whether the subspace F has a basis and whether the constructed sub Γ system is a basic sequence. This question is closely related to the following problem in finite dimensional spaces about which very little is known. Given a finite dimensional space E is there a basis $\{e_1,\cdots,e_n\}$ such that the projections $\sum_1^n \alpha_i e_i \to \sum_1^m \alpha_i e_i$ have norm $\leqq K$ where K is independent of n, m, and E? It is known that such a K, if it exists, will be strictly larger than 1 [2].

The following is not hard. If E has sub Γ system (x_n) and if (x_n) is not a Γ system in $[x_n]$ then $[x_n]$ is not complemented in E. There exist such E and $\{x_n\}$ in an example of Zippin's, with (x_n) basic and $E = c_0$ [18].

The following problems are related to the Hilbert space problem of § 1. If E has noncomplemented subspace do there exist X_n, finite dimensional, such that $\lambda(X_n) \to \infty$? If every subspace with a basis is complemented is the above true? If every subspace is complemented or if every subspace with a basis is complemented is the space reflexive? If E is reflexive with a conditional basis (e_n) is some subsequence (e_{n_k}) a sub Γ or sub Λ system?

One easily sees that if a space fails to have the Grothendieck approximation property, then every complete sequence in the space is at least a Γ system. On the other hand, if (x_n) is complete, and if $y_n = x_n + \sum_1^{n-1} a_{nk} x_k$ is a basis, then (x_n) cannot be a Γ system.

Thus, if E has a basis then it has a complete non Γ system. Conversely, if P_n is a projection from E onto $[x_1,\cdots,x_n]$ and $\| P_n \| \leqq k$ for each n does $[x_n]$ have a basis? This problem has been raised by V. N. Nikolskii [15]; he has shown that under an additional hypothesis the answer is affirmative.

If one can show that every space with basis having no Λ system is isomorphic to Hilbert space, then one solves the Hilbert space problem (§ 1), if the following conjecture of Pelczynski [16] is true: A separable space is isomorphic to Hilbert space if and only if every subspace having a basis is isomorphic to Hilbert space. We show, using Remark 1 after Theorem 1, that if we replace basis by Schauder decomposition into finite dimensional subspaces, then the answer is affirmative:

PROPOSITION. *If E is not isomorphic to Hilbert space, then E has a separable subspace F having a Schauder decomposition into finite dimensional subspaces, and not isomorphic to l_2.*

Proof. Let $E_1 \subset E$, $\dim E_1 < \infty$ and let $F_1 \subset E$ have codim $F_1 < \infty$ such that $E_1 \cap F_1 = \{0\}$ and the natural projection of $E_1 \oplus F_1$ onto E_1 has norm $\leqq 1 + \varepsilon$. F_1 cannot be isomorphic to Hilbert space, since we assumed E is not. Therefore, by a theorem of Dvoretzky [5] and Joichi [9], F_1 contains a subspace E_2 of finite dimension such that $\operatorname{dist}(E_2, l_2^2) \geqq 2$. We can also find $F_2 \subset F_1$ such that codim $F_2 < \infty$, $F_2 \cap E_1 \oplus E_2 = \{0\}$ and the natural projection of $E_1 \oplus E_2 \oplus F_2$ onto $E_1 \oplus E_2$ has norm $\leqq 1 + \varepsilon$. Continuing in this way, one obtains sequences (E_n) and (F_n) with $F_{n+1} \subset F_n$, $E_{n+1} \subset F_n$, $\dim E_n < \infty$, codim $F_n < \infty$, $\operatorname{dist}(E_n, l_2^n) \geqq n$, and such that the natural projection of $E_1 \oplus \cdots \oplus E_n \oplus F_n$

onto $E_1 \oplus \cdots \oplus E_n$ has norm $\leqq 1 + \varepsilon$. Then, $F = \sum_{n=1}^{\infty} E_n = \{ \sum e_n \mid e_n \in E_n$ and $\sum e_n$ converges in $E\}$ is the desired subspace.

In case E has a sub Γ system, then for any Y of finite codimension in E one easily constructs an increasing sequence (U_j) of finite dimensional subspaces such that $[U_j] = Y$ and $\lambda(U_j) \to \infty$. Can one always do this for some Y of infinite codimension? In particular, if (x_n) is a Λ system, is there a subsequence (x_{n_k}) which is a sub Λ system and such that $[x_{n_k}]$ has infinite codimension?

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Ohio State University,

and

Institute of Mathematics of the Romanian Academy of Sciences,

and

Florida State University